Module 1

The Structure of Numbers

Sets and subsets

 Natural numbers

 Partitions

 Primes, Composites, and {1}

 Evens and Odds

 Three Theorems

 Groups

 Definition and Examples

 Some sets and operations that fail to be groups

 The Cayley table for Evens and Odds

Real numbers and their structure

 Rational numbers and one use for the ratio aspect of them

 The Group of Evens and Odds

Complex numbers and Quaternions, a brief introduction

Homework

With this module, we begin the study of numbers. Some of the information will be quite familiar and some will be new. I hope you enjoy the new material and that it informs your intuition and knowledge of numbers for years to come.

We start with the most basic structure in math: sets. A set is a collection of objects. In our class, we will be working with sets of numbers. In a larger sphere, you may talk about sets of just about anything. It is important that a set have a property that is called being “well-defined”. A well-defined set has a definition or set statement that allows a reasonable person to decide whether or not a given object is in the set or not without too much work.

The set of all currently enrolled students at UH is a well-defined set. Given any person in the whole wide world, it only takes looking at a paid up to date fee bill to decide if the person is currently enrolled or not.

The set of natural numbers is well-defined, too. These are the counting numbers. The smallest counting number is 1 and there is no largest set element. We usually denote this set with a capital *N* and may include a partial list of them in set builder notation:

*N* = {1, 2, 3, 4, 5, …}

*F* = {5, 10, 15, 20, …, 5*p*, … where *p* is a natural number} =

{ 5*p*where *p* is a natural number}.

It is important to define your variables. Lower case *n* is often used to mean natural number but it is not guaranteed to be a natural number…always provide a description for your reader. I used “*p*” when discussing *F* above, though and it’s ok because I defined it to be a natural number for you.

A proper subset has a special symbol “ “. We can write *F*  *N* and indicate to a reader that *N* has at least one element that is not in *F*. Sometimes you really don’t know if a subset is proper or not or you really might want to include the original set in your comment. In that case you use “ “ meaning “ is a proper subset or is the original set”.

Popper 1, Question 1

If you are working with a set of numbers, you may want to organize the set according to some properties. In fact, if you organize the whole set into disjoint subsets you have partitioned the original set (disjoint sets share no elements; the intersection of disjoint sets is empty.)

For example, one partition on the natural numbers is:

{{1}, {primes}, and {composites}}.

How do I know it is a partition?

So now I have a collection of 3 subsets and I can drop any given natural number into exactly one set. Did you notice the outside pair of set braces? This is a set of sets!

Where does 15 go? To {composites}. And 41? To {primes}.

Popper 1 Question 2

Now back to {{1}, {primes},{composites}}

This is a very fundamental partition. We will study primes exclusively in an upcoming module.

We can do a little bit with {1} right now. This is a special number with many uses and one important quirk.

We use this fact when finding and using common denominators.

If you need to add: 

you quickly find the least common multiple of the denominators. It is 30. We multiply each term by a carefully chosen one, one at a time.

Look:

 

The number 1 has MANY representations!

We also use this multiplication fact in units analysis when switching from one set of units to another.

Suppose I have 2 yards of fabric. How many inches is that? I will use conversion factors.

1 yard = 3 feet 1 foot = 12 inches

I cannot do much with those equations but if I divide both sides of the first equation by “1 yard” I find that

1 =  another handy one is  = 1 acquired by division, too.

Now 2 yards (1)(1) is definitely still 2 yards. Substitute for those factors of 1 now



We have 2 yards at both ends of this equation. Using a couple of carefully chosen ones has transformed the look of those yards though.

What is another partition on the natural numbers? How about evens and odds?

First let me write out the definitions for these sets:

*E* = {2*n* where n is a natural number} and

*O* = {2*n* – 1 where n is a natural number}.

A partial list of each is:

*E* = { 2, 4, 6, 8, 10,…2*n*, … where *n* is a natural number}

*O* = {1, 3, 5, 7,… 2*n* – 1, …where *n* is a natural number}

Is every natural number even or odd (not both, not neither)? In other words, do these two disjoint proper subsets of the natural numbers union back to *N*?

Aside:

Look at a number line that highlights the naturals and mark off both the evens and the odds along with the remainders from dividing by 2:

Do you see the pattern? We can represent the numbers by their remainders even!

Now we can do one more flourish on this and discuss comparing numbers. I like to talk about this now because we will be using it again this semester.

It turns out that addition and multiplication are similar operations: they create new numbers. Subtraction and division are slightly different: they each compare numbers as well as create new numbers. When you compare with subtraction it is very easy to see which number is bigger. And when you compare with division it is easy to tell which number is bigger.

5 + 3 = 8 5 – 3 = 2 > 0 3 – 5 = −2 < 0

5(3) = 15 5/3 = 1 2/3 > 1 3/5 < 1

Moreover, when you divide two numbers and the remainder is zero, the divisor is a multiple of the dividend! This is valuable information sometimes.

So with evens and odds, we can compare natural numbers to 2 with division. If the remainder is zero, we know the number is even and if the remainder is 1 we know the number is odd. We could even rename E to 0 and O to 1 if we wanted to.

At the end of the course we will revisit this idea of comparing to a given number by division. And we will be looking at partitions throughout the whole semester.

We can look at some theorems about evens and odds and then look where this goes in modern algebra (modern in that this is algebra since about 1850 and not traditional algebra which has been around since before Christ was born, literally).

**Theorem 1**

The sum of two even numbers is even.

**Theorem 2**

The sum of two odd numbers is even.

**Theorem 3**

The sum of an even number and an odd number is odd.

To **illustrate** **Theorem 1**, I take any two even numbers and add them and show that the sum is even. For example 16 and 52 are two even numbers. 16 + 52 = 68. Is 68 even? There are two ways to show that. First, divide 68 by 2 – the remainder is zero. Perfect, 68 is in *E*. The second way is to show that 68 = 2*n* where *n* is a natural number. In this case *n* = 34 and 68 is in *E*.

Notice that the two ways of showing that 68 is even have to do with long division. The first way focuses on the remainder and the second way focuses on the quotient. We have the divisor (2) and the dividend (68) as well.

Now to **prove** **Theorem 1**, I take two even numbers IN FORMULA FORM. I do not choose specific numbers, I choose abstract numbers of the correct type. I will choose 2*p* and 2*q* where *p* and *q* are natural numbers. Notice that I am **not** using *n* twice, I am choosing variables that allow for different even numbers as well as the same number used twice. This ensures that I am really choosing ANY even numbers; we call these “arbitrary even natural numbers”.

So now I will add them then factor out the 2: 2*p* + 2*q* = 2(*p* + *q*)

The big question here is: is (*p* + *q*) a natural number? Which is to say:

Is the sum of two natural numbers another natural number?

The answer is yes and later in this module I will provide a discussion of why it is.

We clearly have fit the pattern, the formula form, of an even number so the proof is done.

Now we can illustrate and prove **Theorem 2**:

The sum of two odd numbers is even.

**Illustration**: 37 + 15 = 52 = 2(26)

**Proof**:

Let 2*p* – 1 and 2*q* – 1 be any two odd numbers with *p* and *q* natural numbers. Note that these are arbitrary odd natural numbers.

Their sum is:

(2*p* – 1) + (2*q* – 1) = 2*p* + 2*q* – 2 = 2(*p* + *q* – 1).

So, even if *p* = *q* = 1, we still have 2 times a natural number in the end.

This fits the pattern for an even number. Done.

Now **Theorem 3**:

The sum of an even number and an odd number is odd.

**Illustration**:

Popper 1 Question 3

**Proof**:

Let 2*p* be any even number and 2*q* – 1 be any odd number with *p* and *q* being natural numbers.

Their sum is:

2*p* + 2*q* – 1 = 2(*p* + *q*) – 1.

The number on the right belongs in *O* because it meets the definition, fits the formula pattern, for an odd number. Done.

Theorem 1: The sum of two even numbers is even

Will become E + E = E (I could also write 0 + 0 = 0)

Theorem 2 will become O + O = E (and 1 + 1 = 0, nonstandard but in context)

And Theorem 3 is E + O = O (and 0 + 1 = 1)

And now I’m going to organize this information into a nice little table:

|  |  |  |
| --- | --- | --- |
| Add |  E |  O |
|  E |  |  |
|  O |  |  |

Let’s do a Cayley Table for the version where E = 0 and O = 1:

|  |  |  |
| --- | --- | --- |
| + | 0 | 1 |
| 0 |  |  |
| 1 |  |  |

Yes it looks odd, doesn’t it? But in this context do you get it?

An algebra is a set and a way to combine set elements:

For example above: (S, +) where S = {E, O} or {0, 1}

An algebra must have the following properties:

closed

identity element.

inverse element

associative property

Now before we analyze the table above, we can take refuge in the traditional algebra of real numbers.

Popper 1 Question 4

Before this course, it may never have occurred to you that there is more than one algebra. Well, there are LOTS of them and the traditional algebra you studied earlier in your life fits the definition of an algebra, too. Let’s check that out.

What will our set be? real numbers. And our combining operation will be addition. We put them together like this (ℜ, +) . This is called a math structure.

Note that you need some context to ensure that nobody thinks I’m talking about some weird point in the Cartesian Plane!

Now if you add two real numbers you get a real number. (This is something a math major would have to prove, but we can take it as a reasonable sensible statement). This is closure.

Aside:

Earlier in this module when proving Theorem 1 for evens and odds, I asked if (*p* + *q*) is a natural number; this is the reason that it is. Closure in the algebra that is ({E,O}, +) or ({0, 1}, +). Back to traditional algebra!

The identity under addition is 0. If you add zero to any real number, no change happens. We call zero the additive identity for this reason.

Each real number has an additive inverse. It is −1 times the original number. And when you add a number and its additive inverse you get 0, the identity.

And the associative property of real number addition holds. An illustration:

 (5 + 3) + 2 = 5 + (3 + 2)

So this is the algebraic structure that underpins all the solving for x and graphing that goes on in an Algebra II class.

As always, when a new idea that is a Big Idea is introduced we can give some examples of sets and combining operations that DON’T work just to highlight what does work!

Looking at (*N*, +) the natural numbers and addition.

What about subtraction and whole numbers?

(*W*, −) The whole numbers include zero.

What about the whole numbers and addition? (W, + )

Closed: check

Identity: check

Inverses: stop

Another try:

Integers include the whole numbers and the negative natural numbers!

(*J*, +)

Now let’s check associativity VERY carefully.

(5 + −3) + −1 is this equal to 5 + (−3 + −1)?

Why am I being so picky about putting in those plus signs?

Hey! This works! Good, here’s another group to go with (ℜ, +)!

Now, an aside.

 (Rationals without zero, multiply) is our newest group.

Closed? Yes!

Identity: 1

Inverses: *a* and …*a* and it’s reciprocal

Associative? Yes!

Now we do not always need to have infinite sets to have an algebra.

Here is one of many finite abstract algebras.

Suppose my set is *F* = {0, 1, 2, 3} and I will define \* to combing set elements in the following way:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  \* |  0 |  1  |  2 |  3 |
|  0 |  |  |  |  |
|  1 |  |  |  |  |
|  2 |  |  |  |  |
|  3 |  |  |  |  |

Isn’t the arithmetic strange? Well let’s look closely at the table:

So this is closed because the entries in the table are only from the set *F*. Which is what makes the arithmetic odd! We WANT closure.

What is the identity element?

What are the inverses – search in the body of the table for the zeros.

Illustrate associativity. (not prove – those are too hard for this level!)

So this is a fairly small finite group.

Now back to evens and odds.

|  |  |  |
| --- | --- | --- |
| Add |  E |  O |
|  E |  |  |
|  O |  |  |

This is a little bitty group. And it is a rather high level of abstraction, too.

E and O have an infinite number of set elements and we are just adding them in the abstract, setwise. We are using our theorems to set up a representation of a LOT of work.

How do we know it is closed?

What is the identity element?

What are the inverses?

Illustrate associativity:

What if I redo the table with 0 and 1?

|  |  |  |
| --- | --- | --- |
|  + |  0 |   1 |
|  0 |  |  |
|  1 |  |  |

More odd arithmetic, but we know what we are saying right?

The sum of two odd natural numbers is even!

What about the properties?

Closed?

Identity?

Inverses?

Associative?

So what we have here is a connection between number theory and algebra. The people who work in each area can discuss and access each other’s thoughts by knowing just a bit about the other field. This is an important part of knowing that the field of mathematics has areas of specialty and that there are connections between these areas.

Hands on arithmetic classes handle problems like 3 + 5 = 8…at a higher level we classify the numbers as even and odd and prove our theorems…then we transition to the abstraction of a group with a Cayley table and represent the theorems very efficiently along with fulfilling the requirements of yet another structure, a group.

We mentioned some familiar subsets of real numbers as we were searching for groups with addition. Whole numbers and integers, we looked at these. There are also rational numbers to look at and irrational numbers.

The subset relationship is this:

Natural numbers  Whole numbers  Integers Rational numbers

An additional disjoint subset is the Irrational numbers.

In fact, the irrational numbers and the rational numbers form a partition on the real numbers.

We can draw the usual set diagram of the real numbers here:

The set of rational numbers is designated as *Q*. The set of irrational numbers is designated *P*. Now *Q* is taken from the word “quotient” and is usually the name of the rational numbers because using *R* would cause confusion with the real numbers since both sets have names that start with the same letter. The joke about *P* is that only someone who is psychic can understand them. On an additional note the prefix “ir” means beside. So the irrational numbers are beside the rational numbers in the number line. And they are! Just to the left or right of any rational number is an irrational number. Check it out:

2.9090090009…< 3 < 3.1010010001…

Popper 1 Question 5

So we have rational numbers and irrational numbers. We will look at rational numbers here and pick up irrational numbers in a later module.

For completeness, though, here are the definitions of each:

*Q* = { *a* is any integer and *b* is any integer but zero}

*P* = {numbers whose decimal representation is nonterminating and nonrepeating}

Taken together, the elements of *Q* and *P* are the set of real numbers.

The ratio of *a* to *b* has many uses in real life and in math. This form, a ratio, is a way to express a part of the whole. It is also a way to give arithmetic instructions. It is a way to represent a place on the number line, too.

Arithmetic instructions: 4/2

4/2 is used as another representation for 2. Note that 2/1 is yet another representation for 2 and it is another way to show that the natural numbers are a proper subset of the rational numbers because we have demonstrated that 2 fits the definition of *Q* above above.

This idea “fits the definition of” is important in math. It helps sort mathematical objects and by doing this students learn to sort all kinds of life objects, too.

Declaring that a statement “is not true” is the result of a sorting process that really matters sometimes! Sorting begins in math classes in elementary school with the projects that have children mark all the things that are “alike” on a worksheet. Sorting is very mathematical behavior.

Now about that “part of the whole idea” – there is some interesting math in that notion.

If our whole is a rectangle and we divide the rectangle into 5 identical smaller pieces, then we select 3 of these to use in another project, we have taken 3/5 of the whole to work with.



A more typical problem on this notion is:

A fifth grade class is using equilateral triangle pattern blocks to represent mixed numbers and fractions.

Given that this arrangement represents 5/8,



sketch an arrangement that represents 5/4 = 1.25 = 125%.

Think for a moment about how you would tackle this problem.

Now let’s do it together:

Here is another one:



This disc represents 3/5. How would you represent 6/15?

Here is a collection of square pattern blocks that represents ¾.

How would you represent 1.25?



None of these problems can be solved without knowledge of traditional algebra and rational numbers.

Popper 1 Question 6

Now, real numbers are not the only numbers available to us to solve problems and define the universe as we know it. Real numbers are a proper subset of complex numbers which are, in turn, a proper subset of quaternions. We will take a brief look at these larger sets and then finish up this module with a review.

Solving more problems creatively is a goal of everyone, in every context!

Suppose you want to solve for *x* with the equation

 

With a little creative work you find that *x* is a new kind of number!

The square root of negative 1: *i*  =

And  is the solution. Imagine a positive and a negative version of this number!

We can sketch a set containment picture for these new sets. Then we can look at the complex numbers more closely.

The set containment illustrates the idea that every real number is a complex number. We generally write complex numbers with a real part, *a*, and an imaginary part, *b*. We signal which number is *b* by having it multiplied by *i*.

A complex number is *a* + *bi* where *a* and *b* are real numbers. And we’re just getting used to that *i* number.

5 + 2*i*

The definition of complex number is a number that is represented as *a* + *bi* where *a* and *b* are real numbers and *i* =.

2 is a complex number because it fits the definition: 2 + 0*i*.

*− i* is a complex number because it fits the definition: 0 − 1*i*.

Let’s look more closely at the set containment picture to see that the reals are a proper subset of the complex numbers.

5

3.1 −0.5 *i*

Let’s write both of these out in the pattern set by the definition above and note that each meets the definition.

Complex numbers were first mentioned in the early 1500’s by an Italian mathematician and geometer Gerolamo Cardano. He was working on finding solutions to cubic equations. They proved pretty handy and mathematicians eventually found many uses for them. So did physicists! If you like MRI technology and light bulbs and electricity, then you have to like complex numbers…they’re present in the math behind many, many modern conveniences!

Now we can look at the Argand Plane.

The Argand Plane, named for Jean-Robert Argand (1768-1822), was also published by a Norwegian mathematician Caspar Wessel (1845 – 1818). It is a geometric representation of a complex number.

If you take a real number line horizontally and a complex number line vertically…oops! So what exactly is a complex number line?

Now, reals horizontally and complexes vertically:

This is the Argand Plane. Is this a plane? What’s the only difference from the Cartesian Plane? What notation might confuse students?

Note that you really need to be clear on your context!

We can find a few purely real numbers and a few purely complex numbers and some with a real part plus a complex part in this plane:

5, −3, *i*, −*i*, 5 + 2*i*, −1 + 3i…

And alternate, context dependent notation is:

(5, 0) (−3, 0) (0, *i*)

Popper 1 Question 7



Now you might think that complex numbers are hugely different from real numbers but they are really only a bit different. They add and multiply…there is an algebra for them, of course.

There are even cyclic numbers in both systems under multiplication! Now you already know a little factlet that might not have struck you as very interesting but it really is fun. In real numbers using multiplication, 1 to any natural number power is 1 and −1 cycles between positive 1 and negative 1. If the exponent on (−1)n is an even natural number the result is positive 1 and if the exponent is an odd natural number, the result is negative one.

On a real number line we get:

We can look at *i* and multiply to see what we get. We need to go to 16 or so for you to see the pattern:

Now we can put this on the Argand Plane and see what it looks like

This is a four step cycle!

Of course, math nerds weren’t content to just notice this and move on! Oh my no!

People quickly figured out an easy way to know which number out of the cycle you have for BIG exponents.

Let’s make a circle diagram and analyze it:

What do you notice about the powers? Do you see that they can be represented as multiples of 4 plus a consistent remainder?

Let’s explore this a bit…with an eye on grouping in groups of 4? Why 4?

So now, what does  simplify to and why did I pick that year?

How would you handle …let’s write out the steps, one by one.

Do you see where and how we are grouping? This is just an application of the grouping explanation of multiplication and division. Let’s look at how…

6 is two groups of 3 or 3 groups of 2. When you divide by 2 you get 3, the number of groups of two. 7 is two groups of 3 plus 1.

When you divide 1927 by 4 you get the number of groups of 4 with the appropriate remainder. Using the groups as powers – each group of 4 is 1 with only a few leftover multiplied *i*s to go.

Let’s convert a simple example to the circle superposed on the axes of the Argand Plane. Use 15 as the exponent…

Circling works well with small exponents but you want to fall back on grouping by four for big exponents.

Now you’d think that we have enough numbers, but math people are fidgety types. In working with three dimensions and geometry…in particular, rotations in 3D, the Irish mathematician William Rowan Hamilton decided that a new type of number would be handy to have. He developed the theory of quaternions and caused a minor revolution in geometry and physics with it. He began publishing in 1843 and even had a volume on the theory of quaternions published posthumously!

So what is a quaternion? Most important to us, it is a number set for which the complex numbers are a proper subset! Let’s draw that.

And what does a quaternion look like?

*a* + *b*i + *c*j + *d*k

Where *a*, *b*, *c*, and *d* are real numbers and the i, j, and k are unit vectors that are orthogonal to one another (at right angles to one another).

Do you see the reals? The complexes? Well, it turns out there is no 3-part useful number so we add on TWO new parts to get quaternions (do you see “quad” for 4 parts in the name?)

What is an example of a quaternion…a really gnarly one is



And what about 5, , −*i*, and 3 – 2*i*? Do they fit the definition?

Now we can do a quick analysis and see exactly where natural numbers are in this Big Picture scheme of things:

Popper 1 Question 8

So now, where are we? Very far in outer number space, right? We started with natural numbers and subsets of natural numbers. We looked at proper subsets. Then we branched out into other sets of numbers, the integers and the real numbers, complex numbers, and quaternions.

We also explored a couple of partitions on the natural numbers and a partition on the real numbers. The notion of a partition will come up again and again. A partition is a collection of disjoint proper subsets of a given set that union back to the original set. In other words each element is placed in exactly one subset; every element goes somewhere in the collection of subsets and there are no intersections of these sets that have elements in them. This is an amazingly useful way to talk about properties of sets.

Along the way we found out that there are many algebras, not just Algebra 2 from high school. We looked at the properties and rules for an algebra. There has to be a set and a combining operation. There must be closure, an identity element, an inverse for each element in our set, and the associative property must hold. We will come back and visit algebras later too.

And we found that there is at least one companion plane to the Cartesian Plane, the Argand Plane. And that we can place complex numbers there nicely.

Lots of what we did will come up again in the next modules. The process of illustrating a theorem and then proving a theorem will be an ongoing theme. The ways to compare numbers by division will come up again. The process of comparing an object to a given definition to see if it fits or not is very mathematical behavior and we will do it often.

This module is just the beginning of a journey that will, I hope, help you gain an overview of algebra and number theory, two “proper subsets” of mathematics.

**Module 1 Homework**

Problem 1 6 points

Given the English alphabet as your original set, list 2 proper subsets of it.

Problem 2 6 points

Given the set of digits, {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} create your own partition on this set. Have at least 3 subsets in your partition.

Problem 3 6 points

Add 

Point out in your work where you are using the fact that one times a number makes no change, i.e. where are you using the fact that one is the multiplicative identity and has many representations?

Problem 4 8 points

Convert 5 cm to miles. Point out in your work where you are using carefully chosen factors of 1. Review scientific notation to report your answer.

**Problem 5 Points: 2;2;8 for 12 points total**

Illustrate Theorems 4 and 5, prove Theorem 6:

Theorem 4: The product of two even numbers is even.

Theorem 5: The product of two odd numbers is odd.

Theorem 6: The product of an even number and an odd number is even.

Illustration of Theorem 4: The product of two even numbers is even.

Illustration of Theorem 5: The product of two odd numbers is odd.

Proof of Theorem 6:

The product of an even number and an odd number is even.

Problem 6 10 points

Here is a set: T = {0, 1, 2} and a Cayley table that illustrates ^, a way to combine the elements of T. (T, ^) is a group. Demonstrate that you can see this by answering the following questions.

|  |  |  |  |
| --- | --- | --- | --- |
|  ^ |  0 |  1 |  2 |
|  0  |  0 |  1 |  2 |
|  1 |  1 |  2 |  0 |
|  2 |  2 |  0 |  1 |

How do you know you have closure?

What is the identity element and why are you sure it IS the identity

Make a list of the inverses

Illustrate associativity with two examples

Problem 7 15 points

Here are 3 pattern block problems to solve. Show your work in agonizing detail. Answers without work are worth only 0.5 points each.

A This shape represents 7/9. Show a shape that represents 1 1/6 using this shape.

B These triangle blocks represent 75% of a whole.

 Show a shape that represents 150% of a whole.



C These blocks represent 2/7. Show the representation for 3/14.



Problem 8 8 points

Write a BRIEF essay on how to simplify ; include why you are doing what you are doing.

Show the answer on a mini-Argand Plane.

One page, front side only, 12 point type.

Problem 9 10 points

Invent a problem to go with this shape. You pick the fractions or mixed numbers; you solve the problem in detail.

The blocks below represents \_\_\_\_\_\_\_. Show a shape that represents \_\_\_\_\_\_\_.



Show your work on how to solve it!